New Method to Reveal the Conflict Between Local Realism and Quantum Mechanics

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We formulate the expectation value of the Bell-Żukowski operator acting on qubit states of a two-particle Bell experiment. By using the equivalence between a set of N copies of a two-qubit experiment and a standard two-setting Bell experiment in an entangled $2N$-particle state, we obtain an inequality, which we may call the Bell-Żukowski inequality. It determines whether the measured correlation functions of two-particle states can be modeled locally and realistically. In this Bell experiment of two particles, the conflict between local realism and quantum mechanics is discussed in conjunction with the violation of the Bell-Żukowski inequality. The main point of the result is that the Bell-Żukowski operator can be represented by the Bell-Mermin operator. The threshold visibility of two-particle interference analyzed in this scheme shows good agreement with the value to cause a violation of the Bell-Żukowski inequality.

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I. INTRODUCTION

Bell inequalities that correlation functions satisfying local realistic theories must obey can be violated by certain quantum predictions, as Bell reported in 1964 [1]. Bell used the singlet state, or EPR pairs [2], to show that the correlation functions measured in such singlet states cannot be modeled by local realistic models. Likewise, a certain set of correlation functions produced by quantum measurements of a quantum state contradicts certain predictions of local realistic theories. Those states also cannot be modeled by local realistic models. Up to now, local realistic theories have been studied extensively [3, 4, 5]. Many experiments have shown that Bell inequalities and local realistic theories are violated [6, 7, 8, 9, 10]. Later, in a work by Fine [11], a set of correlation functions can be described with the property that they are reproducible by local realistic theories for a system in two-partite states if and only if the set of correlation functions satisfies the complete set of (two-setting) Bell inequalities. This result is generalized [12, 13] to a system described by multipartite states in the case where two dichotomic observables are measured per site.

In this paper, we present a method using two Bell operators [14] to refute local realistic models of a quantum state. In order to do so, we need only a two-setting and two-particle Bell experiment reproducible by local realistic theories. Such a Bell experiment also reveals the conflict between local realism and quantum mechanics in the sense that the Bell-Żukowski inequality [15] is violated.

Let us consider two-qubit states that, under specific settings, give correlation functions reproducible by local realistic theories. Imagine that N copies of the states can be distributed among $2N$ parties in such a way that each pair of parties shares one copy of the state. The parties perform a Bell-Greenberger-Horne-Zeilinger (GHZ) $2N$-particle experiment [12, 13, 16] on their qubits. Each of the pairs of parties uses the measurement settings noted above. The Bell-Mermin operator [14, 17], B, for their experiment does not show any violation of local realism. Nevertheless, one can find another Bell operator, which differs from B by a numerical factor, that does show such a violation. That is, the original two-qubit states cannot be modeled by local realistic models.

More specifically, the situation is as follows: A given two-setting and two-particle Bell experiment is reproducible by local realistic theories. However, the experimental correlation functions can compute a violation of the Bell-Żukowski inequality. Therefore, actually measured data reveal that the measured state cannot be modeled by local realistic models. Thus, a conflict between local realism and quantum mechanics is revealed. We can see this phenomenon by the simple algebra presented below.

This phenomenon can occur when the system is in a mixed two-qubit state. We analyze the threshold visibility for two-particle interference to reveal the conflict mentioned above. It is found that the threshold visibility agrees with the value to obtain a violation of the Bell-Żukowski inequality.

II. BELL-MERMIN OPERATOR AND BELL-ŻUKOWSKI OPERATOR

Let $\mathbf{N}_{2N}$ be $\{1, 2, \ldots, 2N\}$. We consider the following specific Bell-Mermin operator (see Eq. (23)):

$$B_{\mathbf{N}_{2N}} = 2^{(2N-1)/2}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|). \tag{1}$$

Here, the states $|\Psi_0^\pm\rangle$ are GHZ states [18], i.e.,

$$|\Psi_0^\pm\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes 2N}\rangle \pm |1^{\otimes 2N}\rangle). \tag{2}$$

An average of the Bell-Mermin operator is evaluated by using a standard two-setting Bell experiment. See Fig. 1.

One can introduce a $2N$-partite Bell operator, which one may call the Bell-Żukowski operator Z_{2N}, which differs from $B_{\mathbf{N}_{2N}}$ only by a numerical factor. The Bell-Żukowski operator Z_{2N} [19] is

$$Z_{2N}=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2N}(|\Psi_0^+\rangle\langle\Psi_0^+|-|\Psi_0^-\rangle\langle\Psi_0^-|). \tag{3}$$

An average of the Bell-Żukowski operator is evaluated by using an all-setting Bell experiment. See Fig. 2.

Clearly, we see that the Bell-Mermin operator given in Eq. (1) is connected to the Bell-Żukowski operator Z_{2N} in the following relation:

$$Z_{2N}=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2N}\frac{1}{2^{(2N-1)/2}}B_{\mathbf{N}_{2N}}. \tag{4}$$

One can see that the specific two-setting Bell $2N$-particle experiment in question computes an average value of the Bell-Żukowski operator $\langle Z_{2N}\rangle$ when an average value of $\langle B_{\mathbf{N}_{2N}}\rangle$ is evaluated. Of course, this argument is due to the validity of quantum mechanics. See Fig. 3.

From the Bell-Żukowski inequality

$$|\langle Z_{2N}\rangle|\le 1, \tag{5}$$

we have a condition on the average of the Bell-Mermin operator $\langle B_{\mathbf{N}_{2N}}\rangle$, which is written by

$$|\langle B_{\mathbf{N}_{2N}}\rangle|\le 2\left(\frac{2}{\pi}\right)^{2N}2^{(2N-1)/2}. \tag{6}$$

Please notice that the Bell-Żukowski inequality $|\langle Z_{2N}\rangle|\le 1$ is derived under the assumption that there are predetermined 'hidden' results of the measurement for all directions in the rotation plane for the system in a state. On the other hand, the Bell-Mermin inequality is derived under the assumption that there are predetermined 'hidden' results of the measurement for two directions for the system in a state. We see that a violation of the condition in Eq. (6) implies a violation of the Bell-Żukowski inequality. Our aim is to compute an expectation value of the Bell-Mermin operator given in Eq. (1) by using a two-particle Bell experiment reproducible by local realistic theories. The Bell-Żukowski inequality is stronger than the standard Bell inequalities for $N\ge 2$. This is why a standard Bell experiment reproducible by local realistic theories reveals the conflict between local realism and quantum mechanics.

III. EXPERIMENTAL SITUATION

We consider the following two-qubit states:

$$\rho_{a,b}=V|\psi\rangle\langle\psi|+(1-V)\rho_{\rm noise}\ (0\le V\le 1), \tag{7}$$

where $|\psi\rangle$ is a Bell state defined as

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|+^a;+^b\rangle-i|-^a;-^b\rangle). \tag{8}$$

$\rho_{\rm noise}=\frac{1}{4}\mathbb{1}$ is the random noise admixture. The value of V can be interpreted as the reduction factor of the interferometric contrast observed in the two-particle correlation experiment. The states $|\pm^k\rangle$ are eigenstates of the z-component of the Pauli observable, σ_z^k, for the kth observer. Here, a and b are the labels of the parties (say Alice and Bob). Then, we have $\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b]=0, \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b]=0, \mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b]=V$, and $\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]=V$.

Here, σ_x^k and σ_y^k are Pauli-spin operators for the x-component and for the y-component, respectively. This set of experimental correlation functions is described with the property that they are reproducible by local realistic theories. See the following relations along with the arguments in Ref. 10:

$$\begin{aligned}
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b]-\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b]+\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b]+\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]|=2V\le 2,\\
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b]+\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b]-\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b]+\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]|=0\le 2,\\
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b]+\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b]+\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b]-\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]|=0\le 2,\\
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b]-\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b]-\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b]-\mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]|=2V\le 2.
\end{aligned} \tag{9}$$

In the following section, we will use this kind of experimental situation. Those experimental correlation functions can compute a violation of the Bell-Żukowski inequality. In order to do so, we shall compute an expectation value of the Bell-Mermin operator in the next section.

IV. CONFLICT BETWEEN LOCAL REALISM AND QUANTUM MECHANICS

Imagine that N copies of the states introduced in the preceding section can be distributed among $2N$ parties in such a way that each pair of parties shares one copy

of the state

$$\rho^{\otimes N} = \underbrace{\rho_{1,2} \otimes \rho_{3,4} \otimes \cdots \otimes \rho_{N-1,N}}_{N}. \tag{10}$$

Suppose that spatially separated $2N$ observers perform measurements on each of $2N$ particles. The decision processes for choosing measurement observables are space-like separated. It can be regarded as a standard two-setting Bell experiment in an entangled state in $2N$ particles. See Fig. 4.

We assume that a two-orthogonal-setting Bell-GHZ $2N$-particle correlation experiment is performed. We choose measurement observables such that

$$A_k = \sigma_x^k, A'_k = \sigma_y^k. \tag{11}$$

Namely, each of the pairs of parties uses measurement settings such that they can check the condition in Eq. (9). Therefore, it should be that given 2^{2N} correlation functions are described with the property that they are reproducible by local realistic theories. The Bell-Mermin operators $B_{\mathbf{N}_{2N}}$ and $B'_{\mathbf{N}_{2N}}$ do not show any violation of local realism as shown below.

Let $f(x,y)$ denote the function

$$\frac{1}{\sqrt{2}} e^{-i\pi/4}(x+iy), x, y \in \mathbf{R}. \tag{12}$$

$f(x,y)$ is invertible as

$$x = \Re f - \Im f, y = \Re f + \Im f. \tag{13}$$

The Bell-Mermin operators $B_{\mathbf{N}_{2N}}$ and $B'_{\mathbf{N}_{2N}}$ are defined by [16, 17]

$$f(B_{\mathbf{N}_{2N}}, B'_{\mathbf{N}_{2N}}) = \prod_{k=1}^{2N} f(A_k, A'_k). \tag{14}$$

The Bell-Mermin inequality can be expressed as [17]

$$|\langle B_{\mathbf{N}_{2N}}\rangle| \le 1, \quad |\langle B'_{\mathbf{N}_{2N}}\rangle| \le 1. \tag{15}$$

We also define B_α for any subset $\alpha \subset \mathbf{N}_{2N}$ by

$$f(B_\alpha, B'_\alpha) = \prod_{k\in\alpha} f(A_k, A'_k). \tag{16}$$

It is easy to see that, when $\alpha, \beta (\subset \mathbf{N}_{2N})$ are disjoint,

$$f(B_{\alpha\cup\beta}, B'_{\alpha\cup\beta}) = f(B_\alpha, B'_\alpha) \otimes f(B_\beta, B'_\beta), \tag{17}$$

which leads to following equations:

$$\begin{aligned} B_{\alpha\cup\beta} &= (1/2)B_\alpha \otimes (B_\beta + B'_\beta) + (1/2)B'_\alpha \otimes (B_\beta - B'_\beta), \\ B'_{\alpha\cup\beta} &= (1/2)B'_\alpha \otimes (B'_\beta + B_\beta) + (1/2)B_\alpha \otimes (B'_\beta - B_\beta). \end{aligned} \tag{18}$$

In specific operators A_k and A'_k given in Eq. (11), where

$$\sigma_x^k = |+^k\rangle\langle -^k| + |-^k\rangle\langle +^k| \tag{19}$$

and

$$\sigma_y^k = -i|+^k\rangle\langle -^k| + i|-^k\rangle\langle +^k|, \tag{20}$$

we have (cf. [20])

$$\begin{aligned} f(A_k, A'_k) &= (e^{-i\frac{\pi}{4}}/\sqrt{2})(\sigma_x^k + i\sigma_y^k) \\ &= e^{-i\frac{\pi}{4}}\sqrt{2}|+^k\rangle\langle -^k| \end{aligned} \tag{21}$$

and

$$\begin{aligned} f(B_{\mathbf{N}_{2N}}, B'_{\mathbf{N}_{2N}}) &= \prod_{k=1}^{2N} f(A_k, A'_k) \\ &= e^{-i\frac{2N\pi}{4}} 2^N \prod_{k=1}^{2N} |+^k\rangle\langle -^k| \\ &= e^{-i\frac{2N\pi}{4}} 2^N |+^{\otimes 2N}\rangle\langle -^{\otimes 2N}|. \end{aligned} \tag{22}$$

Hence, we obtain

$$\begin{aligned} B_{\mathbf{N}_{2N}} &= 2^N\big\{(1/2)(e^{-i\frac{2N\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| + H.c.) \\ &\quad -(-i/2)(e^{-i\frac{2N\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| - H.c.)\big\} \\ &= 2^{\frac{2N-1}{2}}(e^{-i\frac{(2N-1)\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| + H.c.) \\ &= 2^{(2N-1)/2}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|), \end{aligned} \tag{23}$$

where

$$e^{-i\frac{(2N-1)\pi}{4}}|+^{\otimes 2N}\rangle = |1^{\otimes 2N}\rangle. \tag{24}$$

Measurements on each of $2N$ particles enable them to obtain 2^{2N} correlation functions. Thus, they get an average value of the specific Bell-Mermin operator given in Eq. (1). According to Eq. (18), we obtain

$$\langle B_{\mathbf{N}_{2N}}\rangle = \langle B'_{\mathbf{N}_{2N}}\rangle = \prod_{i=2}^{N}\langle B_{\{i-1,i\}}\rangle = V^N(\le 1). \tag{25}$$

Clearly, the Bell-Mermin operators $B_{\mathbf{N}_{2N}}$ and $B'_{\mathbf{N}_{2N}}$ for their experiment do not show any violation of local realism as we have mentioned above.

Nevertheless, when $N \ge 2$ and V is given by

$$\left(2\left(\frac{2}{\pi}\right)^{2N} 2^{(2N-1)/2}\right)^{1/N} < V \le 1, \tag{26}$$

we have a violation of the condition in Eq. (6), i.e., one can compute a violation of the Bell-Żukowski inequality $|\langle Z_{2N}\rangle| \le 1$ that is, the measured two-qubit state cannot be modeled by local realistic models. The condition in Eq. (26) says that the threshold visibility decreases when the number of copies, N, increases. In an extreme situation, when $N \to \infty$, we have the desired condition

$$V > 2(2/\pi)^2 \tag{27}$$

to show the conflict in question. This agrees with the value to get a violation of the Bell-Żukowski inequality.

It is worth noting that the condition in Eq. (27) gives $V > 0.81$, which does not seem to conflict with the condition in Eq. (26).

The given example using two-qubit states reveals the violation of the Bell-Żukowski inequality. The interesting point is that all the information to get the violation of the Bell-Żukowski inequality can be obtained only by a two-setting and two-particle Bell experiment reproducible by local realistic theories.

V. SUMMARY

In summary, we have shown that the Bell-Żukowski operator can be represented by the Bell-Mermin operator. This fact provides a means to check whether a quantum state can be modeled by local realistic models, i.e., if the conflict between local realism and quantum mechanics occurs. Our argument relies only on a two-setting and two-particle Bell experiment reproducible by local realistic theories. Given a two-setting and two-particle Bell experiment reproducible by local realistic theories, one can compute a violation of the Bell-Żukowski inequality. Measured data, thus, indicate that the measured state cannot be modeled by local realistic models. Thus, the conflict between local realism and quantum mechanics is revealed. This phenomenon can occur when the system is in a mixed state. We also analyzed the threshold visibility for two-particle interference in order to bring about the phenomenon. The threshold visibility agrees well with the value to obtain a violation of the Bell-Żukowski inequality.

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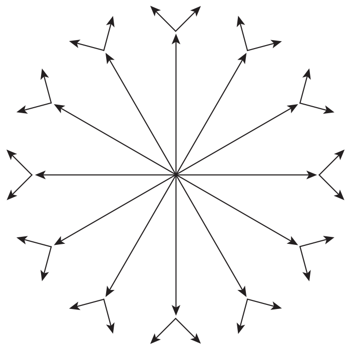

FIG. 1: Schematic diagram of a standard two-setting Bell experiment in an entangled state in twelve particles with the Bell-Mermin operator $B_{\mathbf{N}_{2N}} = 2^{(2N-1)/2}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|)$ acting on Greenberger-Horne-Zeilinger states $|\Psi_0^{\pm}\rangle = (|0^{\otimes 2N}\rangle \pm |1^{\otimes 2N}\rangle)/\sqrt{2}$.

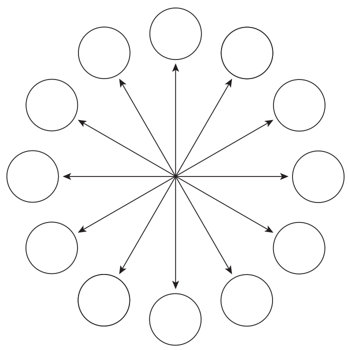

FIG. 2: Schematic diagram of a Bell-Żukowski experiment in an entangled state of twelve particles with the Bell-Żukowski operator $Z_{2N} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2N}\frac{1}{2^{(2N-1)/2}}B_{\mathbf{N}_{2N}}$.

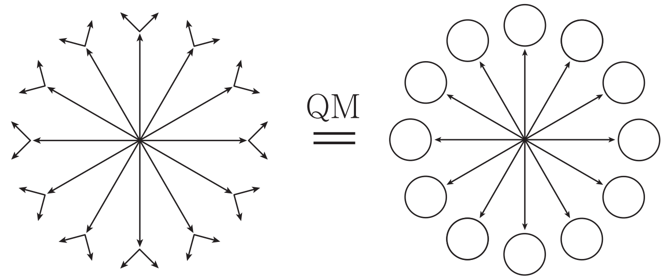

FIG. 3: Schematic diagram of the equivalence between a Bell-Żukowski experiment and a standard two-setting Bell experiment under the validity of quantum mechanics.

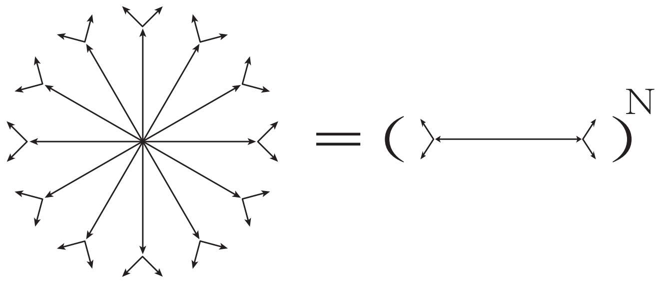

FIG. 4: Schematic diagram of N copies of two-qubit experiments which are equivalent to a standard two-setting Bell experiment in an entangled $2N$-particle state.